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Irreducible Representations of Infinite-Dimensional Lie Algebras

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Some results concerning the operatorially irreducible, topologically irreducible, operatorially and topologically irreducible representations of a class of infinite-dimensional Lie algebras by bounded operators in a complex Banach space are obtained.

Let $\mathscr{A}$ be a Lie algebra. $\mathscr{A}$ is quasisolvable [8, 13] if $\mathscr{A} = \sum_{\alpha \in \Lambda} \mathscr{I}_\alpha$, $\mathscr{I}_\alpha$ is a finite-dimensional solvable ideal of $\mathscr{A}$ for any $\alpha \in \Lambda$. Obviously, a finite-dimensional solvable Lie algebra is quasisolvable; *a quasisolvable Lie algebra may be infinite dimensional.* In [9] we have proved that any completely irreducible representation of a quasisolvable Lie algebra by bounded operators on a complex Banach space $\mathscr{X}$ is of dimension $\leqslant 1$. (A representation of group or algebra in Banach space $\mathscr{X}$ is called completely irreducible if its associative hull is weakly dense in the algebra of all bounded operators on $\mathscr{X}$.) In [4] Gurarii gives a partial extention of this result proving that each operatorially and topologically irreducible representation of a finite-dimensional Lie algebra by bounded operators on a complex Banach space is finite dimensional. (A representation T of group or algebra in Banach space $\mathscr{X}$ is called topologically irreducible if there are no closed invariant to T subspaces of $\mathscr{X}$, different from 0 and $\mathscr{X}$. A representation T of group or algebra in Banach space $\mathscr{X}$ is called operatorially irreducible if its commutant $\{T\}'$ in the algebra $\mathscr{B}(\mathscr{X})$ of all bounded operators on $\mathscr{X}$, consists of scalar operators.)

In [10, 11], we have proved some results concerning the topologically irreducible representations of a quasisolvable Lie algebra in a complex Banach space (such a representation by decomposable operators is one-spectral; such a representation by scalar generalized operators has the dimension $\leqslant 1$). For the concepts of decomposable operator, scalar generalized operator and related matters we refer to [3]. A representation T

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is one-spectral if the spectrum of any operator of T contains only one point. A partial extension of these results for topologically irreducible representations of a finite-dimensional Lie algebra is given in [12].

In what follows, we shall prove an extension of the results of [10, 11], for a class of a Lie algebras which may be of infinite dimension. As a particular case we obtain an extention of the above-mentioned results of [9] and [4].

1. Basic Definitions, Examples

In what follows, the dimension of the vector spaces is arbitrary; the cases when the dimension is finite will be especially mentioned. We recall some usual concepts and results. If $\mathscr{A}$ is a Lie algebra and $\mathscr{I} \subset \mathscr{A}$, we denote (see [2]),

$$[\mathscr{I}, \mathscr{I}] = \mathscr{I}' = \mathscr{I}^{(1)}, \qquad \mathscr{I}^{(n)} = [\mathscr{I}^{(n-1)}, \mathscr{I}^{(n-1)}].$$

A Lie algebra $\mathscr{A}$ is solvable when $\mathscr{A}^{(n)} = 0$ for a certain n. An ideal $\mathscr{P}$ of $\mathscr{A}$ is primitive if the following implication is true:

$$\mathscr{I} \text{ ideal in } \mathscr{A}, \qquad [\mathscr{I}, \mathscr{I}] \subset \mathscr{P} \Rightarrow \mathscr{I} \subset \mathscr{P}.$$

A primitive ideal of $\mathscr{A}$ contains any solvable ideal of $\mathscr{A}$. A Lie algebra is semisimple if it contains no nonzero solvable ideals. If any ideal of a Lie algebra is primitive, then the Lie algebra is semisimple (indeed, a primitive ideal of a Lie algebra $\mathscr{L}$ contains any solvable ideal of $\mathscr{L}$; but every ideal of $\mathscr{L}$ is primitive, in particular $\{0\}$ is primitive, hence any solvable ideal of $\mathscr{L}$ is equal to $\{0\}$). If $\mathscr{L}$ is a semisimple Lie algebra of finite dimension, then every ideal $\mathscr{I}$ of $\mathscr{L}$ is semisimple and $\mathscr{I} = \mathscr{I}'$; hence every ideal of a finite-dimensional semisimple Lie algebra is primitive.

Let $\mathscr{G}$ be a Lie algebra and θ a Lie algebra morphism of $\mathscr{G}$. The Lie algebra $\theta(\mathscr{G})$ is semisimple iff $\ker \theta$ is a primitive ideal of $\mathscr{G}$. Indeed, the following implications hold:

$\mathscr{T}$ solvable ideal of $\theta(\mathscr{G}) \Rightarrow$ there exists $n \in N$, so that $[\theta^{-1}(\mathscr{T})]^{(n)} \subset \ker \theta$
$\mathscr{I}$ ideal of $\mathscr{G}$, $[\mathscr{I}, \mathscr{I}] \subset \ker \theta \Rightarrow \theta(\mathscr{I})$ is a commutative ideal of $\theta(\mathscr{G})$.

Definition 1. A morphism θ of a Lie algebra $\mathscr{G}$ is semisimple if $\theta(\mathscr{G})$ is a semisimple Lie algebra (equivalently $\ker \theta$ is a primitive ideal of $\mathscr{G}$).

If any ideal of $\mathscr{G}$ is primitive, then any morphism of $\mathscr{G}$ is semisimple; every morphism of a finite-dimensional semisimple Lie algebra is semisimple. Every finite-dimensional semisimple representation of a Lie algebra is copletely reductive (Weyl's theorem).

Now we define a class of Lie algebras; our purpose is to describe their irreducible representations in a complex Banach space.

DEFINITION 2. A Lie algebra $\mathscr{A}$ is called LM-decomposable if we have $\mathscr{A} = \mathscr{R} + \mathscr{G}$, $\mathscr{R} = \sum_{\alpha \in \Lambda} \mathscr{I}_\alpha$, $\mathscr{I}_\alpha$ is finite-dimensional solvable ideal of $\mathscr{A}$ for any $\alpha \in \Lambda$ and $\mathscr{G}$ is a Lie algebra so that any ideal of $\mathscr{G}$ is primitive.

Obviously, every finite-dimensional Lie algebra is LM-decomposable (Levi–Malcev theorem); every quasisolvable Lie algebra is LM-decomposable ($\mathscr{G} = \{0\}$). Another example of an LM-decomposable Lie algebra is given by an ideally finite Lie algebra.

A Lie algebra $\mathscr{A}$ is called ideally finite [7] if $\mathscr{A} = \sum_{\alpha \in \Lambda} \mathscr{F}_\alpha$ and $\mathscr{F}_\alpha$ is a finite-dimensional ideal of $\mathscr{A}$ for any $\alpha \in \Lambda$. If $\mathscr{I}_\alpha$ is the solvable radical of $\mathscr{F}_\alpha$, then $\mathscr{I}_\alpha$ is an ideal of $\mathscr{A}$ and $\sigma(\mathscr{A}) = \sum_{\alpha \in \Lambda} \mathscr{I}_\alpha$ is the unique maximal locally solvable ideal of $\mathscr{A}$ [7, Proposition 4.1]. It results, as a corollary of [7, Theorem 6.1], that there exists a Levi subalgebra $\mathscr{G}$ of $\mathscr{A}$, hence $\mathscr{A} = \sum_{\alpha \in \Lambda} \mathscr{I}_\alpha + \mathscr{G}$, where $\mathscr{G}$ is a semisimple Lie algebra. Obviously, $\mathscr{G}$ is ideally finite and [7, Theorem 4.8] shows that $\mathscr{G}$ is a direct sum of finite-dimensional simple ideals. Hence every ideal of $\mathscr{G}$ is primitive (if $\mathscr{I}$ is an ideal of $\mathscr{G}$ we have $\mathscr{I} = \mathscr{I}'$, because $\mathscr{I}$ is direct sum of finite-dimensional simple ideals). Hence, an ideally finite Lie algebra is LM-decomposable.

Remark. Clearly $\theta(\mathscr{A})$ is LM-decomposable for every LM-decomposable Lie algebra $\mathscr{A}$ and every Lie algebra morphism θ of $\mathscr{A}$.

The following is similar to the concept of quasisolvable Lie algebra.

DEFINITION 3. A Lie algebra $\mathfrak{N}$ is quasinilpotent if $\mathfrak{N} = \sum_{\alpha \in \Lambda} \mathscr{I}_\alpha$, where $\mathscr{I}_\alpha$ is a finite-dimensional ideal of $\mathfrak{N}$ and $adx \mid \mathscr{I}_\alpha$ is nilpotent for any $x \in \mathfrak{N}$ and $\alpha \in \Lambda$.

Obviously, the following assertions are equivalent:

(n_0) $\mathfrak{N}$ is a quasinilpotent Lie algebra,

(n_1) $\mathfrak{N} = \sum_{\alpha \in \Lambda} \mathscr{I}_\alpha$, $\mathscr{I}_\alpha$ are finite-dimensional nilpotent ideals of $\mathfrak{N}$,

(n_2) $\mathfrak{N}$ is ideally finite and every finite-dimensional ideal of $\mathfrak{N}$ is nilpotent,

(n_3) $\mathfrak{N}$ is ideally finite and $adx \mid \mathscr{I}$ is nilpotent for every $x \in \mathfrak{N}$ and every finite-dimensional ideal $\mathscr{I}$ of $\mathfrak{N}$.

A finite-dimensional quasinilpotent Lie algebra is nilpotent; a quasinilpotent Lie algebra is quasisolvable (hence LM-decomposable); a homomorphic image of a quasinilpotent Lie algebra is quasinilpotent. It is easy to prove

THEOREM 1. *If* $\mathfrak{N}$ *is a quasinilpotent Lie algebra and* $\mathscr{J} \subset \mathfrak{N}$ *is a finite-dimensional ideal of* $\mathfrak{N}$, *then there exist* $\mathscr{J}_0, \mathscr{J}_1, \mathscr{J}_2, \ldots, \mathscr{J}_n$ such that $0 = \mathscr{J}_0 \subset \mathscr{J}_1 \subset \cdots \subset \mathscr{J}_k \subset \cdots \subset \mathscr{J}_n = \mathscr{J}$, $\dim \mathscr{J}_k = k$, $[\mathfrak{N}, \mathscr{J}_k] \subset \mathscr{J}_{k-1}$ *for every* $k = 1, \ldots, n$.

Proof. $x \to adx \mid \mathscr{J}$ is a morphism of Lie algebras. Because $\dim \mathscr{J} < +\infty$, it follows that $\{adx \mid \mathscr{J};\ x \in \mathfrak{N}\}$ is a finite-dimensional nilpotent Lie algebra and the proof results by virtue of Engel's classical theorem.

2. THE IRREDUCIBLE REPRESENTATIONS OF AN LM-DECOMPOSABLE LIE ALGEBRA BY BOUNDED OPERATORS ON A BANACH SPACE

We begin with a characterisation of the operatorially irreducible representations (particularly completely irreducible) of quasinilpotent Lie algebras by bounded operators on a complex Banach space.

Let $\mathscr{X}$ be a complex Banach space, $\mathscr{B}(\mathscr{X})$ the Lie algebra of all bounded operators on $\mathscr{X}([A, B] = AB - BA)$. We denote by $\mathbb{C}$ the field of complex numbers and by I the identity on $\mathscr{X}$.

THEOREM 2. *Let* $\mathfrak{N} \subset \mathscr{B}(\mathscr{X})$ *be a quasinilpotent Lie subalgebra of* $\mathscr{B}(\mathscr{X})$. *If the center of* $\mathfrak{N}$ *is scalar* ($Z_{\mathfrak{N}} = \{\lambda(z) I\}_{z \in Z_{\mathfrak{N}}}$, $\lambda(z) \in \mathbb{C}$), *then* $\mathfrak{N}$ *is scalar.*

Proof. If $Z_{\mathfrak{N}} = \{0\}$, then $\mathfrak{N} = \{0\}$ by virtue of Engel's theorem mentioned before. Let now $Z_{\mathfrak{N}} = \{\lambda I\}_{\lambda \in \mathbb{C}}$. Obviously, for any finite-dimensional ideal $\mathscr{J}$ of $\mathfrak{N}$, $\mathscr{T} = \mathscr{J} + Z_{\mathfrak{N}}$ is a finite-dimensional solvable ideal of $\mathfrak{N}$. There exists by [5], a linear form $\chi: \mathscr{T} \to \mathbb{C}$ and $\{\xi_n\} \subset \mathscr{X}$, $\|\xi_n\| = 1$, so that

$$\lim_{n \to \infty} (r\xi_n - \chi(r)\xi_n) = 0 \qquad \text{for every } r \in \mathscr{T}. \tag{1}$$

Obviously, $\chi \mid [\mathscr{T}, \mathscr{T}] = 0$. Then it follows $\chi \mid Z_{\mathfrak{N}} \cap [\mathscr{T}, \mathscr{T}] = 0$. But $Z_{\mathfrak{N}} = \{\lambda I\}_{\lambda \in \mathbb{C}}$. Hence, $Z_{\mathfrak{N}} \cap [\mathscr{T}, \mathscr{T}] = 0$. We obtain $Z_{\mathfrak{N}} \cap [\mathscr{J}, \mathscr{J}] = 0$ for every finite-dimensional ideal $\mathscr{J}$ of $\mathfrak{N}$, because $[\mathscr{J}, \mathscr{J}] = [\mathscr{T}, \mathscr{T}]$. Finally it follows that,

$$Z_{\mathfrak{N}} \cap [\mathfrak{N}, \mathfrak{N}] = 0. \tag{2}$$

Let $\mathscr{T}_\alpha = \mathscr{J}_\alpha + Z_{\mathfrak{N}}$ (where $\mathfrak{N} = \sum_{\alpha \in \Lambda} \mathscr{J}_\alpha$; see Definition 3). We have by (2),

$$Z_{\mathfrak{N}} \cap [\mathscr{T}_\alpha, \mathfrak{N}] = 0.$$

Then there exists the following decomposition of $\mathscr{I}_\alpha$ in a direct sum of vector spaces:

$$\mathscr{I}_\alpha = Z_{\mathfrak{N}} \oplus \mathscr{I}_{\alpha_1}, \qquad \mathscr{I}_\alpha \supset \mathscr{I}_{\alpha_1} \supset [\mathscr{I}_\alpha, \mathfrak{N}]. \tag{3}$$

The inclusion $\mathscr{I}_{\alpha_1} \supset [\mathscr{I}_\alpha, \mathfrak{N}]$, shows that $\mathscr{I}_{\alpha_1}$ is an ideal of $\mathfrak{N}$; $Z_{\mathscr{I}_{\alpha_1}}$, the center of $\mathscr{I}_{\alpha_1}$, is invariant to any derivation of $\mathscr{I}_{\alpha_1}$. But $\mathfrak{N}$ is a quasinilpotent Lie algebra; hence $adx \mid Z_{\mathscr{I}_{\alpha_1}}$ is nilpotent for any $x \in \mathfrak{N}$. If $Z_{\mathscr{I}_{\alpha_1}} \neq 0$, it follows by Engel's classical theorem ($\dim Z_{\mathscr{I}_{\alpha_1}} < +\infty$) that there exists r_1 such that $0 \neq r_1 \in Z_{\mathscr{I}_{\alpha_1}}$ and $adx(r_1) = 0$ for any $x \in \mathfrak{N}$. By (3) we deduce $r_1 \notin Z_{\mathfrak{N}}$; so, we can find r_1 such that $0 \neq r_1 \in Z_{\mathfrak{N}}$ and $r_1 \notin Z_{\mathfrak{N}}$, contradictory. Hence $Z_{\mathscr{I}_{\alpha_1}} = 0$, so that $\mathscr{I}_{\alpha_1} = 0$ by virtue of Engel's theorem. Then, by (3) we have $\mathscr{I}_\alpha = Z_{\mathfrak{N}}$ for every $\alpha \in \Lambda$, so that $\mathfrak{N} = Z_{\mathfrak{N}}$ and the proof is finished.

As a corollary, we obtain

THEOREM 2′. *Each operatorially irreducible representation of a quasinilpotent Lie algebra by bounded operators in a complex Banach space is scalar* ($\rho(x) = \lambda(x)I$, $\lambda(x) \in \mathbb{C}$).

In the following some results will be given concerning the irreducible *LM*-decomposable Lie subalgebras in $\mathscr{B}(\mathscr{X})$; these results lead in an obvious manner to the results concerning the irreducible representations of an *LM*-decomposable Lie algebra by bounded operators in a complex Banach space. The following results will be used in the proofs:

LEMMA 1. *If* $A, B \in \mathscr{B}(\mathscr{X})$, $[A, B] = \lambda B$, $\lambda \neq 0$, *then* B *is nilpotent.*

The proof is well known (see, e.g., [9]).

LEMMA 2. *If* $\mathfrak{A}$ *is a Lie subalgebra of* $\mathscr{B}(\mathscr{X})$, $\mathscr{C}$ *is an ideal of* $\mathfrak{A}$ *and* $\mathscr{N} = \{Q \in \mathscr{C} \mid Q \text{ nilpotent}\} \subset Z_{\mathscr{C}}$ (*the center of* $\mathscr{C}$), *then* $\mathscr{N}$ *is an ideal of* $\mathfrak{A}$ (*in particular, if* $\mathscr{C}$ *is commutative, then* $\mathscr{N}$ *is an ideal of* $\mathfrak{A}$).

Proof. Because $\mathscr{N}$ is commutative it follows that $\mathscr{N}$ is a vector space. For $X \in \mathfrak{A}$ and $Q \in \mathscr{N}$, we have

$$[X, Q] = A, \qquad [A, Q] = 0,$$

because $\mathscr{C}$ is an ideal and $\mathscr{N} \subset Z_{\mathscr{C}}$. Then, by [11, Propositions 1], it follows that A is nilpotent.

LEMMA 3. *Let* $\mathfrak{A}$ *be a Lie subalgebra of* $\mathscr{B}(\mathscr{X})$ *and* $\mathscr{I}$ *a finite-dimensional ideal of* $\mathfrak{A}$, *so that* $adX \mid \mathscr{I}$ *is nilpotent for every* $X \in \mathfrak{A}$. *Then, one of the following assertions is true*:

(i) $[Q, \mathcal{I}] = 0$ *for every* Q *nilpotent*, $Q \in \mathfrak{A}$,

(ii) *there exists a* $Q_{\mathcal{I}}$ *nilpotent*, $0 \neq Q_{\mathcal{I}} \in Z_{\mathfrak{A}} \cap \mathcal{I}$.

This lemma is [11, Proposition 2].

PROPOSITION 1. *Let* $\mathfrak{A}$ *be a Lie subalgebra of* $\mathcal{B}(\mathcal{H})$ *and* $\{\mathcal{I}_n\}$ *an increasing sequence of ideals of* $\mathfrak{A}$, $\dim \mathcal{I}_n = n$ *for every* n; *then for every* n, *one of the following assertions is true*:

(j) $adX \mid \mathcal{I}_n$ *is nilpotent for every* $X \in \mathfrak{A}$,

(jj) *there exists an* $\mathcal{N}$, $0 \neq \mathcal{N} \subset [\mathfrak{A}, \mathcal{I}_n]$, *a commutative ideal of* $\mathfrak{A}$ *so that every* $A \in \mathcal{N}$ *is a nilpotent operator*,

(jjj) *there exists a* Q, $0 \neq Q \in \mathcal{I}_n \cap Z_{\mathfrak{A}}$, Q *nilpotent*.

Proof. The proof is an adaptation of the proof of [11, Proposition 3]. The conclusion is true for $n = 1$. Indeed, if $A_1 \in \mathfrak{A}$, $\mathbb{C}A_1 = \mathcal{I}_1$, then $[X, A_1] = \lambda(X)A_1$, $\lambda(X) \in \mathbb{C}$ for any $X \in \mathfrak{A}$. If (j) and (jjj) are not true for $n = 1$, there exists $X \in \mathfrak{A}$ so that $\lambda(X) \neq 0$. Hence A_1 is nilpotent in virtue of Lemma 1 and $\mathcal{N} = \mathcal{I}_1$ verifies (jj).

Now let us suppose the conclusion of the proposition true for n. It will be proved that it is true for $n + 1$. The following case remains to be studied: (j) holds for $\mathcal{I}_n$, (jj) and (jjj) are not true for any k, $1 \leqslant k \leqslant n$. In this case, let A_{n+1} be so that $\mathcal{I}_n + \mathbb{C}A_{n+1} = \mathcal{I}_{n+1}$. For any $X \in \mathfrak{A}$ we can write,

$$[X, A_{n+1}] = B_n + \lambda_{n+1}(X)A_{n+1}, \qquad B_n \in \mathcal{I}_n, \ \lambda_{n+1}(X) \in \mathbb{C}. \tag{4}$$

The equality $\lambda_{n+1}(X) = 0$, for any $X \in \mathfrak{A}$, shows that (j) holds for $\mathcal{I}_{n+1}$. Only one case remains to be studied, namely, the case when there exists $X_0 \in \mathfrak{A}$ so that $\lambda_{n+1}(X_0) \neq 0$. Here, from (4) and by the induction hypothesis we deduce

$$0 \neq (adX_0)^{n+1}(A_{n+1}) = \lambda_{n+1}(X_0)(adX_0)^n(A_{n+1}) \notin \mathcal{I}_n.$$

Hence, by Lemma 1, $A'_{n+1} = (adX_0)^n(A_{n+1})$ is nilpotent. Obviously, $0 \neq A'_{n+1} \notin \mathcal{I}_n$. Because (jjj) is not true for $\mathcal{I}_n$, one obtains by Lemma 3 the following implication:

$$Q \in \mathfrak{A},\ Q \text{ nilpotent} \Rightarrow Q \text{ commutes with } \mathcal{I}_n.$$

Particularly, A'_{n+1} commutes with $\mathcal{I}_n$, hence A'_{n+1} commutes with $\mathcal{I}_{n+1}$ ($\mathcal{I}_{n+1} = \mathcal{I}_n + \mathbb{C}A'_{n+1}$, because $A'_{n+1} \notin \mathcal{I}_n$). It follows that Q commutes with A'_{n+1} for any $Q \in \mathcal{I}_{n+1}$, Q nilpotent. The above implication shows that Q commutes with $\mathcal{I}_n$. Hence

$$Q \in \mathcal{I}_{n+1},\ Q \text{ nilpotent} \Rightarrow Q \text{ commutes with } \mathcal{I}_{n+1}.$$

Then,

$$\mathcal{N} = \{Q \mid Q \in [\mathfrak{A}, \mathcal{I}_{n+1}],\ Q \text{ nilpotent}\} \subset Z_{[\mathfrak{A}, \mathcal{I}_{n+1}]}.$$

Therefore $0 \neq A'_{n+1} \in \mathcal{N}$, hence $\mathcal{N} \neq 0$. From Lemma 2, we deduce that $\mathcal{N}$ is a commutative ideal of $\mathfrak{A}$, because $[\mathfrak{A}, \mathcal{I}_{n+1}]$ is an ideal of $\mathfrak{A}$. Hence we found $\mathcal{N}$, $0 \neq \mathcal{N} \subset [\mathfrak{A}, \mathcal{I}_{n+1}]$ a commutative ideal of $\mathfrak{A}$, consisting of nilpotent operators; (jj) is true for $n+1$ and the proposition is proved.

As a corollary we have

PROPOSITION 2. *Let $\mathcal{R} \subset \mathcal{B}(\mathcal{X})$ be a quasisolvable Lie subalgebra of $\mathcal{B}(\mathcal{X})$, i.e., $\mathcal{R} = \sum_{\alpha \in \Lambda} \mathcal{I}_\alpha$ for solvable ideals $\mathcal{I}_\alpha$. Then one of the following statements is true for any $\alpha \in \Lambda$:*

(I) *$adX \mid \mathcal{I}_\alpha$ is nilpotent for every $X \in \mathcal{R}$,*

(II) *there exists $\mathcal{N}$, $0 \neq \mathcal{N} \subset [\mathcal{R}, \mathcal{I}_\alpha]$, $\mathcal{N}$ is a commutative ideal of $\mathfrak{A}$, consisting of nilpotent operators,*

(III) *there exists Q, Q nilpotent, $0 \neq Q \in \mathcal{I}_\alpha \cap Z_{\mathcal{R}}$.*

Proof. As a consequence of Lie's classical theorem, for every $\alpha \in \Lambda$ we can find

$$0 = \mathcal{I}_0^\alpha \subset \mathcal{I}_1^\alpha \subset \cdots \subset \mathcal{I}_k^\alpha \subset \cdots \subset \mathcal{I}_{n_\alpha}^\alpha = \mathcal{I}_\alpha,$$

$\mathcal{I}_k^\alpha$ ideal of $\mathcal{R}$, $\dim \mathcal{I}_k^\alpha = k$, for every k and the proof is a result of Proposition 1.

Let now $\mathfrak{A}$ be an LM-decomposable Lie subalgebra of $\mathcal{B}(\mathcal{X})$. Let the decomposition of $\mathfrak{A}$ be $\mathfrak{A} = \mathcal{R} + \mathcal{S}$, $\mathcal{R} = \sum_{\alpha \in \Lambda} \mathcal{I}_\alpha$ as in Definition 2. We denote,

$$\mathfrak{A}_\alpha = ad\mathfrak{A} \mid \mathcal{I}_\alpha = \{adX \mid \mathcal{I}_\alpha ; X \in \mathfrak{A}\}, \qquad \mathcal{R}_\alpha = ad\mathcal{R} \mid \mathcal{I}_\alpha .$$

Because $\dim \mathfrak{A}_\alpha < +\infty$, $\dim \mathcal{R}_\alpha < +\infty$ and $\mathcal{R}_\alpha$ is a solvable ideal of $\mathfrak{A}_\alpha$, it is well known that the nil-radical of the associative algebra generated by $\mathfrak{A}_\alpha$ contains $[\mathfrak{A}_\alpha, \mathcal{R}_\alpha]$ (see [2, Corollary 2, Theorem 8, Section 5, Chap. II]). Then it is easy to show that $[\mathcal{I}_\alpha, \mathfrak{A}]$ is a finite-dimensional nilpotent ideal of $\mathfrak{A}$. We denote,

$$\mathcal{B}_\alpha = [\mathcal{I}_\alpha, \mathfrak{A}] \subset \mathcal{I}_\alpha$$

and let η be the semisimple representation of $\mathcal{S}$ given by $g \to adg \mid \mathcal{I}_\alpha$ (by virtue of Definition 2 every ideal of $\mathcal{S}$ is primitive, hence every morphism of $\mathcal{S}$ is semisimple). The Lie algebra $\{adg \mid \mathcal{I}_\alpha ;\ g \in \mathcal{S}\}$ is of finite dimension and semisimple. According to Weyl's theorem, $\mathscr{B}_\alpha$ splits into the direct sum,

$$\mathscr{B}_\alpha = \mathscr{B}^0_\alpha + \mathscr{B}^1_\alpha.$$

$\mathscr{B}^0_\alpha$, $\mathscr{B}^1_\alpha$ vector spaces which are invariant to adg for any $g \in \mathscr{G}$, $adg \mid \mathscr{B}^0_\alpha = 0$ for any $g \in \mathscr{G}$ and $\bigcap_{g\in\mathscr{G}} \ker(adg \mid \mathscr{B}^1_\alpha) = 0$.

Let Z_α be the center of $\mathscr{B}_\alpha$. By Lemma 3 one of the following statements is true:

(1) $Q \in Z_\alpha$ for every $Q \in \mathscr{B}_\alpha$, Q nilpotent,

(2) there exists Q nilpotent, $0 \neq Q \in Z_\alpha$.

Obviously there are two possibilities: $\mathscr{B}^1_\alpha \neq 0$ or $\mathscr{B}^1_\alpha = 0$. If $\mathscr{B}^1_\alpha \neq 0$, then Lemma 1 shows that there exists in $\mathscr{B}^1_\alpha$ nonzero nilpotent operators. In this case, (1) or (2) shows that there exists nonzero nilpotent operators in the center of $\mathscr{B}_\alpha$. Obviously Z_α is an ideal of $\mathfrak{A}$ and by Lemma 2 we deduce that

$$\mathscr{N}_\alpha = \{Q \in Z_\alpha \mid Q \text{ nilpotent}\}$$

is an ideal o $\mathfrak{A}$, $\mathscr{N}_\alpha \neq 0$. Hence $\mathscr{B}^1_\alpha \neq 0$ implies the following assertion:

there exists a finite-dimensional commutative nonzero ideal of $\mathfrak{A}$ *consisting of nilpotent operators.* (N)

If $\mathscr{B}^1_\alpha = 0$, we have $\mathscr{B}_\alpha = \mathscr{B}^0_\alpha$; hence,

$$[\mathscr{G}, [\mathfrak{A}, \mathscr{I}_\alpha]] = 0. \tag{5}$$

since $\mathscr{R}$ is quasisolvable, there follows by Proposition 2 one of the following statements:

(a) $adX \mid \mathscr{I}_\alpha$ is nilpotent for every $X \in \mathscr{R}$,

(b) there exists $\mathscr{N}$, a comutative ideal of $\mathscr{R}$, consisting of nilpotent operators, $0 \neq \mathscr{N} \subset [\mathscr{R}, \mathscr{I}_\alpha]$,

(c) there exists Q nilpotent, $0 \neq Q \in Z_{\mathscr{R}} \cap \mathscr{I}_\alpha$.

If (a) holds we deduce by Lemma 3 one of the following assertions:

(a_1) $[Q, \mathscr{I}_\alpha] = 0$ for any $Q \in \mathscr{R}$, Q nilpotent,

(a_2) there exists Q nilpotent, $0 \neq Q \in Z_{\mathscr{R}} \cap \mathscr{I}_\alpha$.

Hence $\mathscr{B}^1_\alpha = 0$ implies one of the assertions (a_1), (b), (c). If (b) holds, we deduce by (5) that $\mathscr{N}$ is an ideal of $\mathfrak{A}$; hence we have (N). If (c) holds, we deduce by Lemma 2 that $0 \neq \{Q \mid Q$ nilpotent, $Q \in Z_{\mathscr{I}_\alpha}\}$ is a finite-dimensional ideal of $\mathfrak{A}$; hence we have (N). We obtain for a fixed $\alpha \in \Lambda$, two possibilities: either (N) holds, or $[\mathscr{G}, [\mathfrak{A}, \mathscr{I}_\alpha]] = 0$, $\mathscr{R}$ verifies (a_1), $adX \mid \mathscr{I}_\alpha$ is nilpotent for any $X \in \mathscr{R}$. Then, for $\mathfrak{A}$ we have the following two possibilities: assertion (N) holds, or for any $\alpha \in A$ we have $[\mathscr{G}, [\mathfrak{A}, \mathscr{I}_\alpha]] = 0$, $\mathscr{R}$ verifies (a_1) and $adX \mid \mathscr{I}_\alpha$ is nilpotent for any $X \in \mathscr{R}$. In the second case we can eliminate the situation when $\mathscr{R}$ contains nonzero nilpotent operators. Indeed, by (a_1) for any $\alpha \in A$ we can find $\mathscr{I} = \sum_{k=1}^{p} \mathscr{I}_{\alpha_k}$ a finite-dimensional ideal of $\mathfrak{A}$ so that $Z_{\mathscr{R}} \cap \mathscr{I}$ contains nonzero nilpotent operators and by Lemma 2,

$$0 \neq \{Q \mid Q \text{ nilpotent}, Q \in Z_{\mathscr{I}}\}$$

is a finite-dimensional ideal of $\mathfrak{A}$, hence (N) holds. If $\mathscr{R}$ contains no nonzero nilpotent operators we obtain $[\mathscr{G}, \mathscr{I}_\alpha] = 0$ for any $\alpha \in A$. Indeed, we can write by virtue of the semisimplicity of the representation $g \to adg \mid \mathscr{I}_\alpha$, $g \in \mathscr{G}$, $\mathscr{I}_\alpha = \mathscr{I}_\alpha^0 + \mathscr{I}_\alpha^1$, $adg \mid \mathscr{I}_\alpha^0 = 0$ for any $g \in \mathscr{G}$ and $\bigcap_{g \in \mathscr{G}} \ker(adg \mid \mathscr{I}_\alpha^1) = 0$. Since $\mathscr{R}$ contains no nonzero nilpotent operators, by Lemma 1 we deduce $\mathscr{I}_\alpha^1 = 0$, so $[\mathscr{G}, \mathscr{I}_\alpha] = 0$ for any $\alpha \in A$. We obtain

THEOREM 3. *If $\mathfrak{A} \subset \mathscr{B}(\mathscr{X})$ is an LM-decomposable Lie subalgebra of $\mathscr{B}(\mathscr{X})$, and $\mathfrak{A} = \mathscr{R} + \mathscr{G}$ is a decomposition of $\mathfrak{A}$, then one of the following statements is true*:

(I) *there exists $\mathscr{N}$, a finite-dimensional commutative nonzero ideal of $\mathfrak{A}$, consisting of nilpotent operators,*

(II) *$[\mathscr{G}, \mathscr{R}] = 0$, $\mathscr{R}$ is a quasinilpotent Lie algebra and $\mathscr{R}$ contains no nonzero nilpotent operators.*

In particular, this theorem is true for ideally finite Lie algebras. On the other hand, we obtain in case (I) a closed subspace of $\mathscr{X}$ which is invariant to $\mathfrak{A}$: $0 \neq \mathscr{X}_0 = \bigcap_{Q \in \mathscr{N}} \ker Q \neq \mathscr{X}$, $\mathscr{X}_0 = \overline{\mathscr{X}}_0$, $A\mathscr{X}_0 \subset \mathscr{X}_0$ for any $A \in \mathfrak{A}$. Clearly we obtain the following irreducible variant of Theorem 3:

THEOREM 3′. *If $\mathfrak{A} \subset \mathscr{B}(\mathscr{X})$ is an LM-decomposable Lie subalgebra of $\mathscr{B}(\mathscr{X})$ which is topologically irreducible and $\mathfrak{A} = \mathscr{R} + \mathscr{G}$ is a decomposition, then $\mathscr{R}$ is a quasinilpotent Lie algebra, $\mathscr{R}$ contains no nonzero nilpotent operators, and $[\mathscr{G}, \mathscr{R}] = 0$.*

By Theorems 3′ and 2 we deduce the following theorem which contains [9, Theorem 1; 4, Theorem 2].

THEOREM 4. *If $\mathfrak{A} \subset \mathscr{B}(\mathscr{X})$ is an LM-decomposable, topologically and operatorially irreducible Lie subalgebra of $\mathscr{B}(\mathscr{X})$, then $\mathfrak{A} = \mathscr{G}$ or $\mathfrak{A} = \mathbb{C}I + \mathscr{G}$, where $\mathscr{G}$ is a semisimple Lie algebra.*

Proof. Let $\mathfrak{A} = \mathscr{R} + \mathscr{G}$ be a decomposition of $\mathfrak{A}$ given by Definition 2. Because $\mathfrak{A}$ is topologically irreducible, we deduce by Theorem 3′, $[\mathscr{G}, \mathscr{R}] = 0$ and $\mathscr{R}$ is a quasinilpotent Lie algebra. Obviously $Z_{\mathscr{R}} \subset Z_{\mathfrak{A}}$. But $\mathfrak{A}$ is operatorially irreducible, hence $Z_{\mathscr{R}} = 0$ or $Z_{\mathscr{R}} = \mathbb{C}I$. By Theorem 2 we obtain $\mathscr{R} = 0$ or $\mathscr{R} = \mathbb{C}I$, because $\mathscr{R}$ is a quasinilpotent Lie algebra. This theorem is true for an ideally finite Lie algebra; when $\dim \mathfrak{A} < +\infty$, this theorem is [4, Theorem 2].

3. Topologically Irreducible Representations of *LM*-Decomposable Lie Algebras by Bounded Operators with Special Spectral Properties

In what follows, the operators with special spectral properties will be decomposable operators, scalar generalized operators [3] and compact operators.

THEOREM 5. *Let $\mathfrak{A} \subset \mathscr{B}(\mathscr{X})$ be an LM-decomposable Lie subalgebra of $\mathscr{B}(\mathscr{X})$, and $\mathfrak{A} = \mathscr{R} + \mathscr{S}$ a decomposition of $\mathfrak{A}$. If $\mathscr{R}$ contains only decomposable operators, then one of the following assertions is true:*

(I) *there exists a nonzero commutative finite-dimensional ideal of $\mathfrak{A}$, consisting of nilpotent operators,*

(II_1) *there exists an $A \in \mathscr{R}$ with nontrivial maximal spectral spaces which are invariant to $\mathfrak{A}$,*

(II_2) *$[\mathscr{S}, \mathscr{R}] = 0$; $\mathscr{R}$ is a quasinilpotent Lie algebra, $\mathscr{R}$ contains no nonzero nilpotent operators and $\mathscr{R}$ is one spectral.*

We recall that a subset $\mathfrak{M} \subset \mathscr{B}(\mathscr{X})$ is one-spectral if the spectrum of any operator of $\mathfrak{M}$ contains only one point.

Proof. Assertion (II) of Theorem 3, Section 2, splits into (II_1) and (II_2) by virtue of Proposition 3.2, Chapter I and Lemma 4.3, Chapter II of [3], because the operators of $\mathscr{R}$ are decomposable. Obviously, the topologically irreducible variant of this theorem is clear.

The nilpotence property of the quasinilpotent scalar generalized operators leads to

THEOREM 6. *Let $\mathfrak{A} \subset \mathscr{B}(\mathscr{X})$ be an LM-decomposable Lie subalgebra of $\mathscr{B}(\mathscr{X})$, and $\mathfrak{A} = \mathscr{R} + \mathscr{S}$ a decomposition of $\mathfrak{A}$. If the center of $\mathscr{R}$ contains only scalar generalized operators, then one of the following assertions is true:*

(I) *there exists a nonzero commutative finite-dimensional ideal of $\mathfrak{A}$, consisting of nilpotent operators,*

(II_1) *there exists $A \in Z_{\mathscr{R}}$ with nontrivial maximal spectral spaces which are invariant to $\mathfrak{A}$,*

(II'') *$\mathscr{R}$ is scalar ($\mathscr{R} = \{\lambda(r)I\}_{r \in \mathscr{R}}$, $\lambda(r) \in \mathbb{C}$).*

Proof. By virtue of Theorem 3, Section 2, it remains to study the case when assertion (II) is true. We have $0 \neq Z_{\mathscr{R}} \subset Z_{\mathfrak{A}}$ or $\mathscr{R} = 0$ because $\mathscr{R}$ is a quasinilpotent Lie algebra and $[\mathscr{S}, \mathscr{R}] = 0$. If $\mathscr{R} \neq 0$, there are two possibilities:

(a) there exists an $r \in Z_{\mathscr{R}}$ so that the spectrum $\sigma(r)$ contains at least two points,

(b) $\sigma(r) = \{\lambda(r)\}$, $\lambda(r) \in \mathbb{C}$ for any $r \in Z_{\mathscr{R}}$.

If (a), we have (II_1) as in the proof of the preceding theorem. If (b), we can write

$$r = \lambda(r)I + Q_r; \qquad \lambda(r) \in \mathbb{C},\, Q_r \text{ nilpotent operator, for any } r \in Z_{\mathscr{R}}.$$

If there exists an $r \in Z_{\mathscr{R}}$ with $Q_r \neq 0$, then we have (I). Hence it remains to study the case when $Z_{\mathscr{R}} = \{\lambda(r)I\}_{r \in Z_{\mathscr{R}}}$. But in this case Theorem 2, Section 2 shows that $\mathscr{R} = \{\lambda(r)I\}_{r \in \mathscr{R}}$, $\lambda(r) \in \mathbb{C}$; hence we have (II″).

The irreducible variant of this theorem is

THEOREM 6′. *Let $\mathfrak{A} \subset \mathscr{B}(\mathscr{X})$ be an LM-decomposable Lie subalgebra of $\mathscr{B}(\mathscr{X})$. If the operators of $Z_{\mathscr{R}}$ (the center of the solvable part of $\mathfrak{A}$) are scalar generalized operators and $\mathfrak{A}$ is topologically irreducible, then $\mathfrak{A} = \mathscr{S}$ or $\mathfrak{A} = \mathbb{C}I + \mathscr{S}$, where $\mathscr{S}$ is a semisimple Lie algebra.*

Finally we shall study the case when $\mathfrak{A}$ contains a nonzero compact operator. Let $\mathfrak{A} \subset \mathscr{B}(\mathscr{X})$ be an *LM*-decomposable Lie subalgebra of $\mathscr{B}(\mathscr{X})$, so that,

$$\mathscr{K}_{\mathfrak{A}} = \{K \in \mathfrak{A} \mid K \text{ compact operator}\} \neq 0.$$

Let $\mathfrak{A} = \mathscr{R} + \mathscr{S}$ be a decomposition of $\mathfrak{A}$. We shall analyse the case when $\mathscr{R}$ is not scalar. By Theorem 3, Section 2, we have (I) or (II). If we have (II), by Theorem 2, Section 2, we deduce that $Z_{\mathscr{R}}$ is not scalar. But $[\mathscr{S}, \mathscr{R}] = 0$; hence $Z_{\mathscr{R}} \subset Z_{\mathfrak{A}}$ and there exists a nonscalar $A_0 \in Z_{\mathfrak{A}}$. Particularly, A_0 commutes with a nonzero compact operator because $0 \neq \mathscr{K}_{\mathfrak{A}} \subset \mathfrak{A}$. We have proved

THEOREM 7. *Let $\mathfrak{A} \subset \mathscr{B}(\mathscr{X})$, be an LM-decomposable Lie subalgebra of $\mathscr{B}(\mathscr{X})$, and let $\mathfrak{A} = \mathscr{R} + \mathscr{S}$ be a decomposition of $\mathfrak{A}$. If $\mathfrak{A}$ contains a nonzero compact operator, then one of the following assertions is true:*

(1) *$\mathfrak{A} = \mathscr{S}$ or $\mathfrak{A} = \mathbb{C}I + \mathscr{S}$, where $\mathscr{S}$ is a semisimple Lie algebra.*

(2) *$0 \neq \mathscr{R} \neq \mathbb{C}I$ and one of the following statements is true:*

(2_1) *there exists a nonscalar $A_0 \in Z_{\mathfrak{A}}$, such that A_0 commutes with a nonzero compact operator,*

(2_2) *there exists a commutative nonzero finite-dimensional ideal of* $\mathfrak{A}$, *consisting of nilpotent operators.*

The irreducible variant of this theorem is an easy consequence of Lomonosov's theorem [6].

THEOREM 7'. *Let* $\mathfrak{A} \subset \mathscr{B}(\mathscr{X})$ *be an LM-decomposable Lie subalgebra of* $\mathscr{B}(\mathscr{X})$ *with a decomposition* $\mathscr{R} + \mathscr{S}$. *If* $\mathfrak{A}$ *contains a compact nonzero operator and* $\mathfrak{A}$ *is topologically irreducible, then* $\mathfrak{A} = \mathscr{S}$ *or* $\mathfrak{A} = \mathbb{C}I + \mathscr{S}$, *where* $\mathscr{S}$ *is a semisimple Lie algebra and in addition* $\mathscr{R} = \{0\}$ *or* $\mathscr{R} = \mathbb{C}I$.

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